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COMMENT

Evolution criterion and renormalisation group for DLA

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Abstract. A renormalisation group method is presented to derive the fractal dimension for the diffusion-limited aggregation (DLA). In a non-equilibrium steady state the dissipation energy and the conductivity of the surface layer play the roles of the free energy and the coupling constant in an equilibrium state. The renormalisation group transformation of the conductivity of the surface layer is obtained in the replacement of the coupling constant. The renormalisation group equation has a non-trivial solution where the derivative at the fixed point has a positive value of less than one. This is consistent with the theorem of minimum entropy production. The fractal dimension is expressed in terms of the derivative of the renormalisation group equation at the fixed point. The fractal dimension $d_f = 1.628$ is found by using the 2×2 small cell renormalisation.

Recently, there has been increasing interest in the physical mechanisms governing the structure in the diffusion-limited aggregation (DLA) (Witten and Sander 1981, Meakin 1983, Family and Landau 1984, Stanley and Ostrowsky 1985, Pietronero and Tosatti 1986, Stanley 1986, Meakin *et al* 1987, Hayakawa *et al* 1987). It is well known that they have a strong measure of self-similarity, which is characterised by the fractal dimension D (Mandelbrot 1982). Several analytical attempts, including mean-field theories (Muthukumar 1983, Tokuyama and Kawasaki 1984, Matsushita *et al* 1986) and position space renormalisation group methods (Gould *et al* 1983, Green 1984, Nakanishi and Family 1985, Kolb 1987), have been made to derive the fractal dimension. Halsey *et al* (1986) and Amitrano *et al* (1986) found the multifractal structure of the growth probability distribution in the surface layer. Nagatani (1987) presented a real space renormalisation group method to derive the multifractality.

Since there was no Hamiltonian formulation of DLA, one adopted the position space renormalisation group approach in which the change in connectivity of the cluster, upon repeated length rescaling, was determined (Gould *et al* 1983, Kolb 1987).

In contrast to the theorem of minimum free energy in an equilibrium state, the non-equilibrium steady states are characterised by an extremum principle according to which the entropy production has its minimum value for a purely dissipative system without convection (Glansdorff and Prigogine 1971). The dynamics of the formation of DLA can be assumed to be determined by a 'quasistationary' diffusion field (Ball 1986). The process of the formation of DLA is a purely dissipative system without convection in a non-equilibrium steady state. The process is characterised by the theorem of minimum entropy production or minimum dissipation energy. We consider the dissipation energy in contrast to the free energy (or Hamiltonian).

In this comment, under renormalisation transformation, we consider the dissipation energy which is represented by the function of the electric field and the conductivity of the surface layer. Under the renormalisation group transformation, the conductivity

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of the surface layer plays the role of coupling constants. We derive the renormalisation group equation of the conductivity of the surface layer. We find the fractal dimension in terms of the derivative of the renormalisation group equation at the fixed point.

We refer to the dielectric breakdown description of DLA. In the discrete version, the dissipation energy is given by

$$H_d = \sum_i \sigma_i E_i^2$$
$$= \sigma_t E_0^2 \tag{1}$$

where the first expression represents the summation of dissipation energy on each bond over total bonds within the system, and the second indicates the total dissipation energy in the representation of which σ_t and E_0 are respectively the total conductance and the total voltage drop of the system. Thus the dissipation energy form (1) describes the interaction between neighbouring potentials by this simple gradient term E_i. The total current, carrying through the surface layer of the aggregate, is given by $J_t = \Sigma_{i_s \in S} \sigma_{i_s} E_{i_s}$ where the sum indicates the summation over all the bonds on the surface layer. We can write the dissipation energy in terms of the surface conductivity:

$$H_d = E_0 \left(\sum_{i_s \in S} \sigma_{i_s} E_{i_s} \right) \tag{2}$$

where σ_{i_s} indicates the conductivity of the bonds on the perimeter of the aggregate. By using a decimation method, one may write the dissipation energy in terms of the coarse-grained variables σ'_{i_s}, E'_{i_s} on the new lattice with the lattice spacing b (b is the scale factor):

$$H_d = E_0 \left(\sum_{i_s \in S'} \sigma'_{i_s} E'_{i_s} \right). \tag{3}$$

Thus the short-range interaction between neighbouring potentials by the gradient term is renormalised. The surface conductivity σ is transformed to σ'. This surface conductivity plays the role of the so-called coupling constant in the Hamiltonian of phase transition. We can obtain the renormalisation group equation:

$$\sigma' = R(\sigma). \tag{4}$$

The renormalisation group transformation (4) of the surface conductivity has been found by Nagatani (1987) from a different point of view. Figure 1 shows a schematic behaviour of the renormalisation function (4). This has a non-trivial solution σ^* (>1). At the fixed point σ^* the derivative $dR/d\sigma$ has a positive value less than one. The equation (4) has a stable fixed point. We here consider the physical meaning of the fixed point. After many repeated renormalisations, the surface conductivity approaches the value σ^* at the fixed point. This is proportional to the total conductance σ_t. Under the boundary condition of the constant voltage ($E_0 =$ constant), we consider a perturbation of the dissipation energy. The variation of the total dissipation energy is given by

$$\delta H_d \sim \delta \sigma_t \sim (\partial \sigma_n / \partial \sigma_{n-1})(\partial \sigma_{n-1} / \partial \sigma_{n-2}) \ldots (\partial \sigma_1 / \partial \sigma_0) \delta \sigma_0. \tag{5}$$

Since $\sigma_n = R(\sigma_{n-1})$ and $0 < (\partial \sigma_n / \partial \sigma_{n-1}) < 1$, the dissipation energy approaches to the minimum value. The theorem of minimum entropy production is consistent with the condition that the RG equation (4) has a stable fixed point.

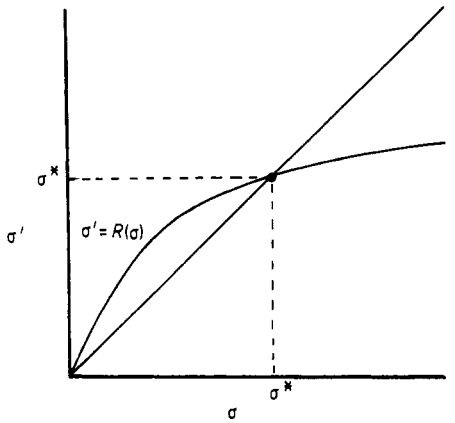

Figure 1. A schematic behaviour of the renormalisation function: $\sigma' = R(\sigma)$. This has a non-trivial solution $\sigma^*(>1)$. At the fixed point σ^* the derivative $dR/d\sigma$ has a positive value less than one. The fixed point is stable.

We derive the expression of the fractal dimension D in terms of the derivative of the RG equation at the fixed point. The number M of particles constituting DLA scales as

$$M \sim r^D \tag{6}$$

where r is the radius of gyration. The variation of the current is proportional to that of the mass:

$$\delta J \sim \delta M. \tag{7}$$

From (6) and (7), we obtain

$$r \sim (\delta J / \delta r)^{1/(D-1)}. \tag{8}$$

The variation of the current is also proportional to that of the surface conductivity:

$$\delta J \sim \delta \sigma. \tag{9}$$

Under the scale transformation $r = br'$, we obtain

$$D = 1 - \ln(\partial\sigma'/\partial\sigma)^*/\ln b \tag{10}$$

where $(\partial\sigma'/\partial\sigma)^*$ indicates the derivative at the fixed point.

We here consider the RG transformation of the surface conductivity for DLA on the square lattice. We cover all the space of the square lattice by cells of edge b (scale factor), each containing $2b^2$ bonds; an example for $b = 2$ is shown in figure 2. After

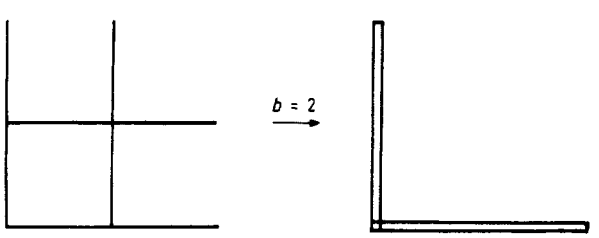

Figure 2. Illustration of the dividing and rescaling of a $b = 2$ cell for DLA on the square lattice.

a renormalisation transformation, these cells play the role of 'renormalised' bonds. The renormalised bonds are then classified into three types of bond: (*a*) break bonds which construct the aggregate; (*b*) growth bonds which are on the perimeter of the aggregate and can be successively grown; (*c*) unbroken bonds which surround the aggregate, except for the growth bonds. The renormalisation procedure is indicated by figure 3. If the cell is spanned with the break bonds then the renormalised bond is considered to be broken (figure 3(*a*)). If the cell is not spanned with the break bonds and is the nearest neighbour to a cell with spanning cluster, then the cell is renormalised as the growth bond (figure 3(*b*)). When the cell is constructed by unbroken bonds only and is not the nearest neighbour to the cells with spanning clusters, the cell is renormalised as the unbroken bond (figure 3(*c*)). We are concerned with the growth bond constructing the surface layer of the aggregate. Figure 4 shows an example of the renormalisation of a part of the surface layer of an aggregate. The

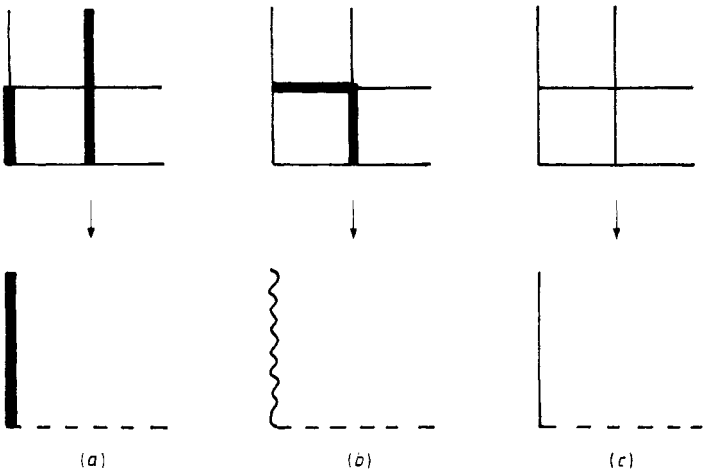

Figure 3. Illustration of the renormalisation of a $b = 2$ cell for DLA. The renormalisation procedure in the vertical direction is shown. There are three types of bonds: break bonds indicated by bold lines, growth bonds by wavy lines and unbroken bonds by light lines. Examples of the distinct configurations are shown in (*a*), (*b*) and (*c*), which are respectively renormalised as break, growth and unbroken bonds.

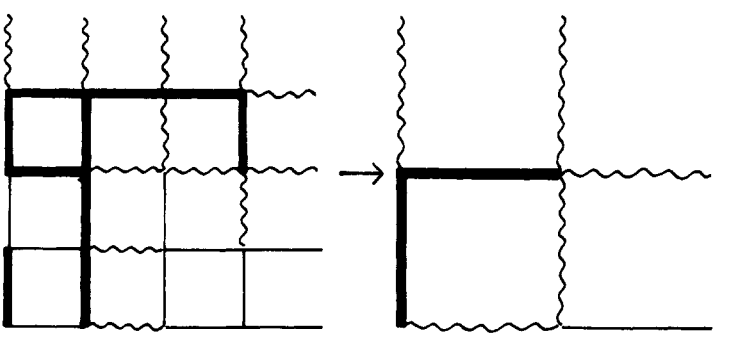

Figure 4. An example of the renormalisation of a part of the surface layer of an aggregate. The lattice on the left-hand side is renormalised to that on the right-hand side, according to the rules of renormalisation.

lattice on the left-hand side is renormalised to that on the right-hand side, according to the rules of renormalisation. We define the conductivity of the growth bond as a surface conductivity. The non-local nature of the electric field is taken into account as the conductivity of the growth bond. We consider the conductivity of the cell that it is possible to renormalise as the growth bond. If we consider the renormalisation in the vertical direction, we shall then take the periodic boundary conditions in a lateral direction. The constant voltage is applied vertically (see figure 5(a)). Figure 5(b) shows all the configurations of the cell for which it is possible to renormalise as the growth bond. Let us consider the configurational probability C_α with which a particular configuration α appears. The distinct configurations are labelled by α ($\alpha = 0, 1, 2$) in figure 5(b). The configuration (1) is constructed by adding a break bond to configuration (0). The probability with which a break bond adds onto the growth bonds 1 or 2 in configuration (0) is given by the growth probabilities $p_{0,1}$ or $p_{0,2}$ of the growth bonds 1 or 2 in the configuration (0). In addition, by adding a break bond to the configuration (1), the configuration (2) occurs. The configurational probabilities C_α are given by

$$\begin{aligned} C_1 &= C_0(p_{0,1}+p_{0,2}) = 2C_0 p_{0,1} \\ C_2 &= C_1(p_{1,2}+p_{1,3}+p_{1,4}) = 3C_1 p_{1,2} \end{aligned} \tag{11}$$

where $p_{0,1} = p_{0,2}$ and $p_{1,2} = p_{1,3} = p_{1,4}$. The configurational probability C_0 is determined from the normalisation condition:

$$\sum_\alpha C_\alpha = C_0 + C_1 + C_2 = 1. \tag{12}$$

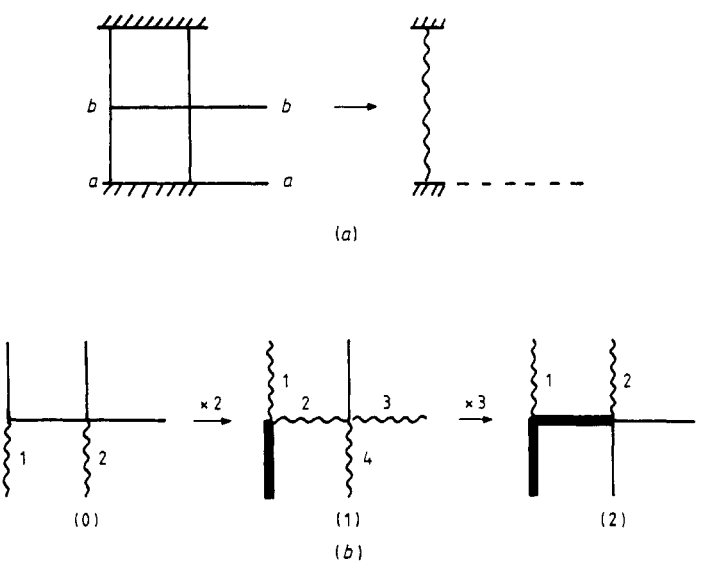

Figure 5. (a) Boundary conditions in the renormalisation in the vertical direction. The constant voltage is vertically applied. The periodic boundary condition is taken in the lateral direction. (b) All distinct configurations of the 2×2 cell that it is possible to renormalise as the growth bond. The configuration (1) is constructed by adding a break bond onto the growth bond 1 or 2 in the configuration (0). Furthermore by adding a break bond onto 2, 3 or 4 in the configuration (1), the configuration (2) is constructed.

The growth probability $p_{\alpha,i}$ on the growth bond i within the cell α is proportional to the electric field E_i on the growth bond. Consider the electrostatic problem for cells which can be renormalised as the growth bond. The electric fields on the growth bonds within a cell are determined by the conductivity of growth bonds and the configuration of the cell. In the configuration labelled by α (see figure 5(b)), the growth probabilities $p_{\alpha,i}$ of growth bonds i are given by

$$\begin{aligned} &p_{0,1} = p_{0,2} = \tfrac{1}{2} \\ &p_{1,1} = (1+3\sigma)/(4+3\sigma) \qquad p_{1,2} = p_{1,3} = p_{1,4} = 1/(4+3\sigma) \\ &p_{2,1} = p_{2,2} = \tfrac{1}{2} \end{aligned} \tag{13}$$

where σ indicates the conductivity of the growth bond, and the conductivities of the break and unbroken bonds are respectively given by infinite value and unit value. The conductivity σ'_α of the cell with configuration α is renormalised as follows:

$$\begin{aligned} &\sigma'_0 = 2\sigma/(1+\sigma) \\ &\sigma'_1 = \sigma + 3\sigma/(1+3\sigma) \\ &\sigma'_2 = 2\sigma. \end{aligned} \tag{14}$$

The renormalised conductivity σ' of the growth bond will be assumed to be given by the configurational average

$$\sigma' = C_0\sigma'_0 + C_1\sigma'_1 + C_2\sigma'_2. \tag{15}$$

The relationships (14) and (15) present the renormalisation group equation $\sigma' = R(\sigma)$. Equations (11)-(15) are simultaneously solved. We find a stable fixed point $\sigma^* = 3.111$ from $\sigma^* = R(\sigma^*)$. From the derivative $(\partial\sigma'/\partial\sigma)^* = 0.647$ at the fixed point, we find the fractal dimension $D = 1.628$ via the relationship (10).

In summary, we present the renormalisation group method to derive the fractal dimension of the diffusion-limited aggregation. We propose that, for the self-similar structure formed by a steady dissipative process, the dissipation energy and the surface conductivity play the role of the free energy and the coupling constant in an equilibrium phase transition. We obtain the RG equation of the surface conductivity. We find that the RG equation has a non-trivial solution (fixed point) where the derivative has a positive value less than one. It is found that the condition $(0 < (\partial\sigma'/\partial\sigma)^* < 1)$ is consistent with the theorem of minimum entropy production. The fractal dimension is found to be expressed in terms of the derivative.

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